Optimising JI guitar designs using linear microtemperaments

or

If it aint Baroque don't waste your lute fixing it

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http://users.bigpond.net.au/d.keenan/Music/MicroGuitar.pdf

Introduction

The easiest alternative tuning systems to implement on a guitar are of course the equal divisions of the octave or equal temperaments (ETs). Some popular alternatives to 12-equal are 17, 19, 22 and 31. Equal divisions are easy because the frets can be continuous across the full width of the neck, and the open strings can be tuned to any note of the scale. In contrast, fretboards for just intonation (JI), extended just intonation and non-equal temperaments such as meantone, typically have many small *fretlets* which are only under one, or a few, strings. This makes it impossible to use the technique of bending notes by pushing strings sideways along the frets, and increases the complexity and cost of the fretting. A 12-note-per-octave scale on such an instrument typically requires the accurate positioning of around 35 fretlets per octave. The method of this paper can significantly reduce this sort of fretboard complexity .

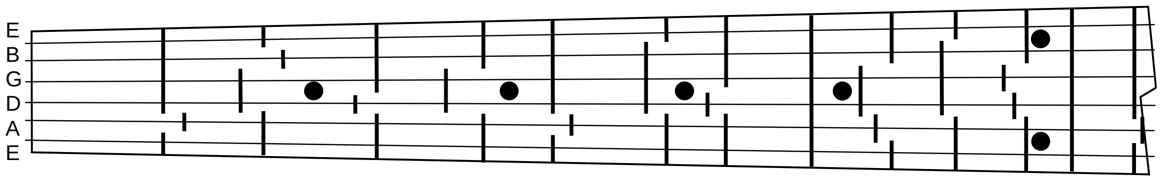


Figure 1. Typical 12-note 5-limit just intonation fretboard

I have tried not to assume any great mathematical sophistication on the part of the reader, but I do assume you are familiar with the use of both ratios and cents for describing pitches and intervals, and are able to convert from ratios to cents, and can calculate the result of stacking intervals in both ratios and cents, including reducing the result back to the first octave. I also assume that you are able to obtain the prime factorisation of a ratio.

I developed the method after collaborating with Paul Erlich on the well-tempering of his *Shrutar*. You can find out more about the Shrutar by searching on this term in the archives of the Yahoo Group *tuning* at http://groups.yahoo.com/group/tuning/. A brief description of the method of this paper was posted to *tuning* on 3-Aug-2001 in the thread *An Eikosany guitar design* (message 26626).

While this paper will only refer to guitars and octave-based tunings, it should be clear that it can be applied to other fretted instruments and non-octave tunings as well.

Note that this paper will not consider how to obtain the correct physical placement of the frets, since that has been well examined by others. See for example the STRLEN command in Manuel Op de Coul's Scala http://www.xs4all.nl/~huygensf/scala/ or the MICROFRET program from Mark Rankin, http://www.afn.org/~sejic/rankin.html, who also sells kits for interchangeable fretboards and can provide a list of luthiers in the USA who are able to make the necessary modifications. Nor will we consider how to choose the correct gauges for the strings. We will be concerned only with obtaining the necessary pitches in cents relative to the open top string. (In this paper I will use the terms top, bottom, high, low to refer only to pitches, and not to physical placements).

One well known approach to maximising the number of full-width frets in JI tunings is to tune the open strings with alternating perfect fifths (2:3) and perfect fourths (3:4) e.g. D:A:D:A:D:A, or perfect fourths (3:4) and major seconds (8:9) e.g. E:A:D:E:A:D, or some combination such as E:A:E:A:B:E which more closely approaches the standard E:A:D:G:B:E tuning. This makes use of the fact that most tunings of interest have many intervals that remain the same after modulation by one or two fifths or fourths.

One problem with this approach is the fact that many highly consonant chords, if they use all six strings, consist of a stack of thirds, or thirds and fourths, of some kind. Having open string intervals of fifths or seconds, makes many such chords difficult to play because too much of a stretch is required to turn a fifth into a third, or a second into a fourth. This problem is reduced somewhat by placing any fifths between low strings and any seconds between high strings.

However, assuming that we are willing to give up the standard open string tuning, I will show how we can design frettings for many justly intoned scales that
(a) have only (or mostly) full-width frets,
(b) have only as many frets as there are notes in the scale (or a very few more) and
(c) have open strings tuned only by fourths and/or thirds.

This seeming magic is achieved by making use of *linear microtemperaments*, so I will first explain what they are.

What is a linear microtemperament?

A *temperament* is, in general, a system of tuning that, while seeking to retain some of the audible advantages of just harmonies, seeks to overcome some of the limitations of tuning systems that are based strictly on simple whole-number ratios, such as a limited ability to modulate, or an excessive number of distinct pitches, some of which may be very close together.

It does this by distributing certain very small intervals called *commas*, over as many intervals as possible. These commas arise because of the fundamental fact of arithmetic that when you multiply prime numbers together, every different bag of primes gives a different result. You can never arrive at the same result by multiplying two different bags of primes, although you can sometimes get very close.

In the best-known example, involving the primes 2, 3 and 5, a stack (or chain) of four just fifths (2:3), when octave reduced, gives an interval of a Pythagorean major third (64:81).

This differs from a just major third (4:5) by the syntonic or Dydimus comma (80:81 or about 21.5 cents). The temperament called *quarter-comma meantone* distributes each such comma uniformly over four fifths, making them all about 5.4 cents narrower than 2:3, so that a stack of four of them gives a just major third, and we don't need separate pitches for Pythagorean and just major thirds.

Meantone temperament is sometimes described as *quasi*-just, which points out the disadvantage of doing this kind of thing. The further we move the tuning of an interval away from a simple ratio, the less it will possess of that unique perceptual quality we call justness.

Now a *linear* temperament is one in which the entire tuning can be generated by chaining (and octave-reducing) a single interval, called the *generator*. Usually there will only be a single chain of generators, but some linear temperaments have multiple chains equally spaced within the octave. Meantone is a linear temperament. The slightly narrow fifth *generates* a 12-note scale as the sequence Eb:Bb:F:C:G:D:A:E:B:F#:C#:G# (or equivalently, a slightly wide fourth can generate it in the reverse order).

Miracle is a linear temperament having a generator which is an approximately 116.7 cent minor second (between 15:16 and 14:15). It approximates all ratios of odd numbers up to 11 (and their octave-reductions, extensions and inversions) with an error no greater than 3.3 cents. We say it has an 11-limit error of 3.3 cents. This greater accuracy and greater range of prime numbers comes at the cost of requiring longer chains of generators (and therefore more notes) to obtain enough of the desired approximations to simple ratios. So we say we are trading error for *complexity*. One measure of this complexity is the number of generators required to be chained to produce at least one of each of the ratios in the specified odd limit. For example the 5-limit complexity of meantone is 4, while the 11-limit complexity of miracle is 22.

Now a *micro*temperament is a temperament whose errors are so small as to leave its intervals and chords sounding, not merely quasi-just, but just. Of course there is no natural boundary between just and quasi-just. Whether any beats are intrusive depends on many contextual factors such as the timbre, the specific chord or intervals being played and their duration, not to mention the discrimination of the listener. But for our practical purpose of guitar design, we can call a temperament "just" or "micro-" if the error it introduces, relative to the theoretical rational tuning, is significantly less than the typical intonation error of a carefully constructed and carefully played guitar.

Not having the resources to conduct a scientific investigation of guitar intonation accuracy, I have relied on anecdotal evidence from the tuning list, and a few published measurements [Guitar Fretboard Intonation Measurements, Jim Campbell, 1998, http://www.precisionstrobe.com/apps/fretbrd/fretbrd.html], as well as a survey of suitable temperaments, to place the cutoff for microtemperaments at a maximum error of 2.9 cents. It seems that even very careful placement of nut, frets and bridge is prone to errors of about this size, while variations in finger pressure alone can easily cause errors of more than twice this. And indeed, with practice, deliberate variations in finger-pressure can be used, when chords are sustained, to compensate for the systematic errors of the microtemperament. When chords are not sufficiently sustained, listeners cannot hear a mistuning anyway (due to the classical uncertainty principle). There is also a possibility that nonlinear coupling between strings on an acoustic instrument may tend to automatically pull the strings into closer agreement with small whole-number ratios.

By this criterion, the Miracle temperament is a microtemperament for 7-limit tunings, since in this case it has a maximum error of only 2.4 cents. You may decide on a lower error cutoff for your microtemperaments, but this will usually require more complex temperaments, which are less likely to meet the design goals. If however you find you can accept slightly larger errors, some less complex temperaments become available, including Miracle for 9 and 11 limit tunings. Even a 4.3 cent error is still only one quarter of the 5-limit error of 12-equal.

Actually *finding* the linear microtemperaments with the lowest complexity for a given set of ratios, was extremely difficult until very recently. In mid 2001, prompted by the rediscovery of the Miracle temperament, Gene Ward Smith and Graham Breed developed mathematical theory and wrote computer programs to generate many new linear temperaments, and to test them to find potentially useful ones – those having the lowest complexities for a given range of error sizes. Graham Breed has made this extraordinary facility available free online for anyone to use at http://www.microtonal.co.uk/temper/.

Defining a few more terms

A few more terms need explaining before we take a look at a list of the best microtemperaments for use in JI guitar design. We find that some of the best linear temperaments do not consist of a single chain of generators, but of two parallel chains spaced a half-octave apart, or even three parallel chains a third of an octave apart, and so on. These can sometimes be considered as if they are single-chain temperaments that repeat at the fraction of the octave instead of the octave. We refer to this interval of repetition as the *period* of the temperament.

In order to make use of a microtemperament in designing a JI guitar fretboard we also need to know the temperament's *prime mapping*. This tells us how many periods and generators are required to approximate each of the prime numbers involved. This then lets us figure out how many periods and generators approximate any given ratio. It is conventional to use the smallest of the several equivalent generators for a temperament. This standard generator will always be smaller than half the period. For example, the standard meantone generator is the approximately 503 cent fourth, and Graham Breed's program gives its 5-limit prime mapping as [(1, 0), (2, -1), (4, -4)]. This is shorthand for

1:2 is approximated by 1 period and 0 generators.
1:3 is approximated by 2 periods and -1 generators.
1:5 is approximated by 4 periods and -4 generators.

Normally we are thinking in octave-equivalent terms, in which case factors of 2 can be ignored and the number of periods is only important when the period is not the octave, and in that case all that matters is the number of periods *modulo* the number in an octave. So we will express the mapping in a simplified octave-equivalent form below.

(a) The simplest linear microtemperaments

	Period	Gen	Comp	Err	Name	Mapping	Rep ET
5-limit	1 oct	316.99 c	6	1.4 c	kleismic	[6 5]	14/53
7-limit	1 oct	116.59 c	13	2.4 c	miracle	[6 -7 -2]	7/72
9-limit	1 oct	316.76 c	22	2.7 c	catakleismic	[6 5 22]	14/53
11-limit	1/2 oct	183.21 c	30	2.4 c	twin minortones?	[-6 -11 2 3; 1 0 1 0]	11/72
13-limit	1/3 oct	83.02 c	48	2.8 c	triple kleismic?	[-6 -5 2 -3-14; 0 2 2 2 2]	5/72
15-limit	1/3 oct	83.02 c	48	2.8 c	triple kleismic?	[-6 -5 2 -3-14; 0 2 2 2 2]	5/72
1.3.5.9	1 oct	498.24 c	10	0.4 c	schismic	[-1 8]	22/53
1.3.5.9.15	1 oct	498.26 c	10	0.4 c	schismic	[-1 8]	22/53
1.3.5.7.9 .15	1 oct	351.45 c	10	1.9 c	complex neutral thirds?	[2 25 13]	29/99
1.3.5.7.9 .11.15	1 oct	193.24 c	35	2.8 c	semithirds?	[-15 2 5 -22]	19/118

(b) Some slightly more complex linear microtemperaments

	Period	Gen	Comp	Err	Name	Mapping	Rep ET
5-limit	1 oct	387.74 c	8	1.4 c	wuerschmidt	[8 1]	21/65
5-limit	1 oct	442.92 c	9	1.5 c	semisixths?	[7 9]	24/65
7-limit	1 oct	193.87 c	16	1.4 c	semiwuerschmidt?	[16 2 5]	16/99
9-limit	1 oct	351.45 c	25	1.9 c	complex neutral thirds?	[2 25 13]	29/99
11-limit	1/2 oct	83.25 c	34	2.9 c	twin semi-10:11s?	[-6-17-10-15; 0 1 1 1]	5/72
11-limit	1 oct	193.24 c	35	2.8 c	semithirds?	[-15 2 5 -22]	19/118

(c) Some simpler but not-so-micro linear temperaments

	Period	Gen	Comp	Err	Name	Mapping	Rep ET
5-limit	1/2 oct	105.21 c	6	3.3 c	diaschismic	[1 -2; 1 1]	7/80
7-limit	1 oct	271.14 c	11	4.3 c	orwell	[7 -3 8]	19/84
9-limit	1 oct	116.72 c	19	3.3 c	miracle	[6 -7 -2]	7/72
9-limit	1 oct	497.81 c	16	4.3 c	schismic	[-1 8 14]	39/94
11-limit	1/2 oct	216.74 c	30	3.1 c	twin thirds?	[-6 -1 10 -3; 1 1 0 0]	13/72
11-limit	1 oct	116.72 c	22	3.3 c	miracle	[6 -7 -2 15]	7/72
11-limit	1/29 oct	16.74 c	29	3.7 c	29-fold 17-cent?	[0 1 1 1; 17 9 23 13]	2/145
11-limit	1 oct	585.14 c	29	4.1 c	complex aug 4ths?	[-7 -26 -25 3]	39/80
13-limit	1 oct	116.76 c	45	3.7 c	miracle	[6 -7 -2 15 38]	7/72
13-limit	1/2 oct	165.81 c	46	3.9 c	twin 10:11s?	[-3-23-5-22-13; 0 1 1 1 1]	8/58
13-limit	1 oct	475.72 c	42	4.0 c	476-cents?	[4 21 -3 39 27]	23/58
13-limit	1/29 oct	16.51 c	29	4.2 c	29-fold 17-cents?	[0 1 1 1 1; 17 9 23 13 20]	2/145
13-limit	1 oct	476.02 c	45	4.3 c	complex 476-cents?	[4-3 7-3-19-31]	23/58
15-limit	1 oct	475.69 c	42	4.1 c	476-cents?	[4 21 -3 39 27]	23/58
15-limit	1 oct	476.02 c	45	4.3 c	complex 476-cents?	[4-37 -3-19-31]	23/58

Names with question marks may differ from those in the archive of http://groups.yahoo.com/groups/tuning-math and are based on the author's uniform naming system (in progress). You may find tri- instead of triple, hemi- instead of semi-, unidec instead of twin minortones, tiny diesic instead of semisixths, wizard instead of twin thirds, and multiple-29 or mystery instead of 29-fold 17-cents.

Table 1. Linear microtemperaments for various odd limits, courtesy of Graham Breed's online temperament finder.

For example, remembering that the canonical generator for meantone is a slightly wide perfect fourth, the 5-limit meantone mapping is given simply as [-1 -4], meaning that a 1:3 (or octave-equivalently a 2:3 perfect fifth) is approximated by a single generator in reverse (or octave-inverted), and a 1:5 (or octave-equivalently a 4:5 major third) is approximated by four generators in reverse (ignoring octaves).

In some cases, distributing a part of the comma to the octave (tempering the octave), and using a slightly different size of generator, will result in an even lower maximum error for the chosen temperament, or may make it feasible to use a less complex temperament. But that means we can no longer use the "odd-limit" terminology and makes the calculations more complicated, and so tempering of the octave is not considered in this paper.

Table 1 first shows the simplest microtemperaments for various odd limits. Of course, odd-limits aren't everything, even for octave-repeating scales. For example, Erv Wilson's 22 note tuning called *Pascal*, has ratios of 1, 3, 5, 7, 9, 11 and 15, but not 13. Graham Breed's temperament finder can handle these cases too, some results are shown at the end of table 1(a).

Now what do we do with them?

So how do we actually go about using these microtemperaments to optimise our JI fretting? Before we get to that, let's look at a simpler case. We'll leave out, for now, the part where we microtemper a JI scale, and instead look at how to find a fretting for a more familiar linear temperament. We'll design a 12-of-meantone guitar (Eb to G#) that meets our criteria. That is, it should use only 12 full-width frets per octave and have open strings tuned in fourths and/or thirds. I had assumed that such a thing was impossible until I applied the method described below.

But before we even get to that, I'd like to spell out in more detail the conditions that we are trying to achieve. There are two main choices to be decided, (a) the intervals between open strings, and (b) the scale rotation (i.e. the pitch of the open top string), which determines the fret layout. You might think of these as the vertical and horizontal aspects of the fretboard. The specific values chosen for various parameters, such as the maximum and minimum interval between adjacent open strings, is somewhat a matter of taste, but I will set down what I have assumed in the following demonstrations of the method.

Criteria for intervals between open strings

1. In the hope of maximising the number of highly consonant chords that are playable, the interval between adjacent strings should be some kind of third or fourth, preferably from a neutral third to a perfect fourth (9:11 to 3:4, approx 350 to 500 cents). But at a pinch, it can go down to a subminor third or up to an augmented fourth (6:7 to 5:7, approx 267 to 583 cents), however two such very small or very large intervals should not occur next to each other, and the extreme open strings should not span less than five neutral thirds or more than five perfect fourths (approx 1750 to 2500 cents).

2. The intervals between open strings do not all have to be just intervals or their approximations, but more chords are likely to be playable if they are. It is preferable if the open string tuning bears some resemblance to 2:3:4:5:6:7, 3:4:5:6:7:9, 3:4:5:7:9:11, 4:5:6:7:9:11 or 5:7:9:11:13:15. But notice that the actual sequence 5:6:7 is prohibited by the constraint above. It is better to have larger intervals between lower strings and smaller intervals between higher strings, rather than the other way ‘round, because the width of the critical band varies with pitch in that way.

3. Open strings must be tuned to notes of the scale.

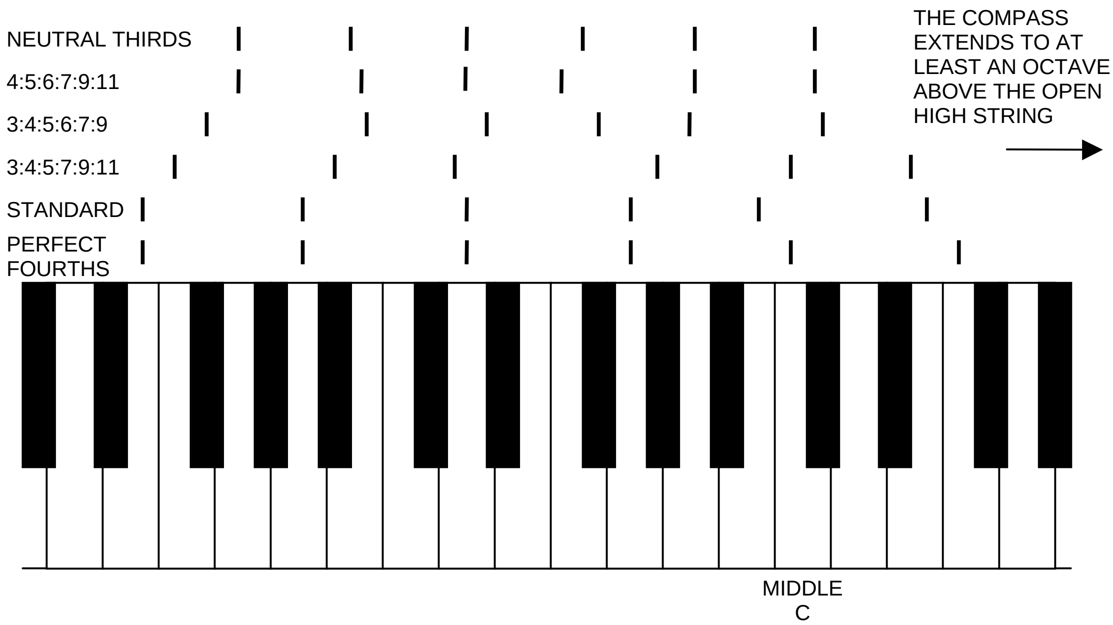


Figure 2. A comparison of some open string tunings

Criteria for scale rotation and fret layout

4. The complete scale must be playable on the high string because the high string has pitches which are not available on any other string. But we do not care which note the high-string-scale starts on (its rotation), except as it affects the other criteria below. We call the high string and any string which is octave-equivalent to it, a *reference string*.

5. There are only as many frets as there are notes in the scale, so reference strings do not have any extra notes outside the scale (*extrascalar* notes).

6. All frets are continuous across the full width of the fretboard, so the fretting of the reference strings is the fretting of all the strings. This means that non-reference strings will (a) be missing some notes of the scale, and (b) have some extrascalar notes.

And finally we come to the crux of the matter.

7. No string should be missing any note of the scale in the region between its open note and the open note of the next higher string. We call this the *critical region* for the string.

(not to be confused with the critical *band* of human pitch perception). Note that we do not require a string to have a fret corresponding to the open note of the next higher string, but this is of course desirable.

It also becomes clear now, why we must have no missing notes *anywhere* on the highest string. Its entire fretted length is critical because there is no “next-higher” string.

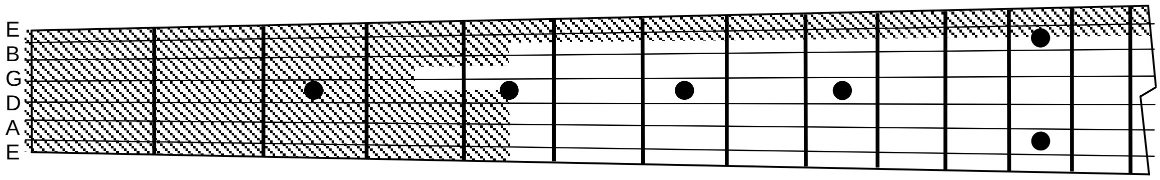


Figure 3. The critical region (shaded) for standard open tuning (12-equal)

Condition 7 (no missing notes in the critical region) ensures that the scale is playable over the entire compass of the instrument, from the open note of the lowest string, to the highest fret of the highest string. In some cases we will find that this cannot be achieved and we will then choose to violate condition 5 and add some extra frets (or fretlets) in the critical region. In this case we will seek to minimise the number of additional fret(let)s required. In other cases we will find that condition 7 is easily achieved, and we will attempt to further optimise the design by placing the first missing note of each non-reference string as *far* as possible outside the critical region.

Even if you decide you must have the complete scale all the way down every string, it will still be advantageous to keep the fretlets as far as possible from the more commonly used region near the nut.

An optimised meantone fretboard

We’ll take as our generator the quarter-comma meantone fourth of 503.4 cents. We can’t assign absolute pitches at this stage, so we will refer to the reference strings as "r" and the non-reference strings as r+1, r-2 etc. where the number gives the string's offset from the reference string along the chain of generators.

When the target scale is contiguous on a single chain of generators, as is our 12-of-meantone, then a string will be missing as many scale notes as the absolute value of its offset from the reference string in generators. So the total number of missing notes (over the whole fretboard) will be minimised by minimising the number of generators by which all the strings are offset from the reference. The hard part is making this happen within the constraints given above, regarding the intervals between adjacent strings and the lack of holes in the critical region.

We need to know which generator offsets correspond to allowed intervals between adjacent strings, so we calculate various multiples of the generator, modulo the octave, and see which are within the range 267 to 583 cents. We also note how many missing notes a string will have for each offset in generators from the reference string.

Interval between open strings

(generators)	-5	**-4**	-3	-2	-1	0	**1**	2	**3**	4	5
(cents)	1083	**386**	890	193	697	0	**503**	1007	**310**	814	117

String designation

(reference+generators)	r-5	r-4	r-3	r-2	r-1	r	r+1	r+2	r+3	r+4	r+5
Number of missing notes	5	4	3	2	1	0	1	2	3	4	5

Figure 4. Meantone generator counts converted to cents. Valid open string intervals are shown in bold. Number of missing notes for each string designation.

We can see that there is no need to depart from the reference string by more than 5 generators since we could, for example, tune them all in fourths. We draw a *state-transition diagram* with circles for the open string tunings, labelled with their offset in generators (the absolute value of which can also be read as the number of missing notes in this case), and arrows for the valid intervals between strings, in the downward pitch direction, labelled with their size in cents. Then we look for sequences of 6 open strings, beginning with r and visiting other states that have as few missing notes as possible. The figure below shows three possibilities in bold. In each case there are only 3 states visited, but each is visited twice to correspond to the 6 strings (ignoring octaves).

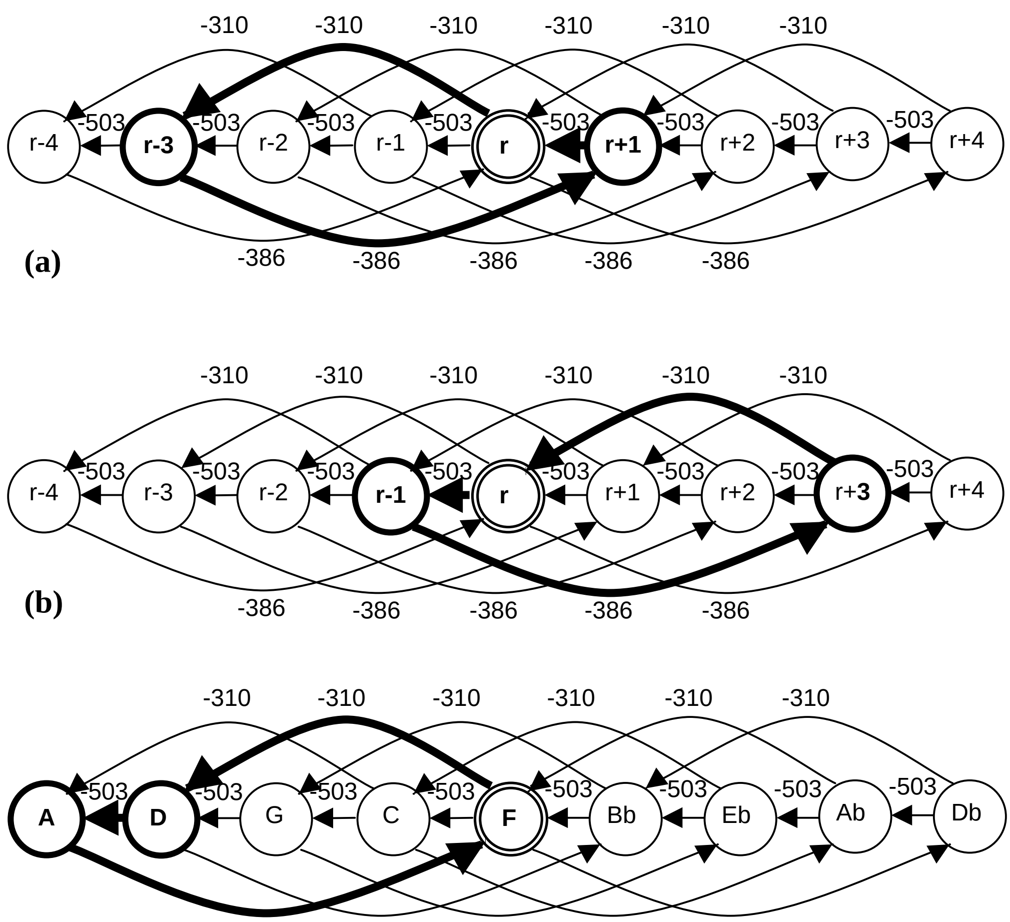


Figure 5. Meantone-12. State transition diagrams showing open string tunings and the intervals between them.

Figure 5(a) shows, descending in pitch, (r) -310 (r-3) -386 (r+1) -503 (r) -310 (r-3) -386 (r+1) for a total span of 1897 cents. This has the advantage of placing the narrower intervals in the higher register and has open major triads.

Figure 5(b) shows, descending in pitch, (r) -503 (r-1) -386 (r+3) -310 (r) -503 (r-1) -386 (r+3) for a total span of 2090 cents. This has the advantage of an extra tone in its compass and has open minor triads.

Figure 5(c) is not optimal in the sense that some strings have more missing notes than any strings in (a) or (b), but it is included out of historical interest (explained below). It shows, descending in pitch, (r) -310 (r-3) -503 (r-4) -386 (r) -310 (r-3) -503 (r-4) for a total span of 2014 cents. It is also included to remind us that there is a less-than-obvious interaction between the choice of open string tuning and the ability to achieve a hole-free critical region. It might conceivably be the case that the latter can sometimes only be achieved by using an apparently sub-optimal open tuning.

The Baroque lute tuning

Historically, figure 5(c) corresponds to the tuning of the highest six courses of the Baroque lute. Ascending in pitch this is A D F A D F, also referred to as the D minor tuning. Wim Hoogewerf kindly pointed this out to me on the Yahoo Group *tuning* when I described 5(b) there. In addition to these six treble courses, a Baroque lute may have up to seven bass courses a tone or semitone apart, but these are usually played open, not fretted. A *course* may consist of a single string or a pair of strings tuned in unison, or in the case of the bass courses, in octaves. This was the final standard open tuning of the lute in Europe, obtained in France in the mid 17th century after about a century and a half of increasing popularity of the instrument, and maintained during the following century and a half of its decline. Thus the use of this open tuning occurred during the meantone era for keyboard instruments.

Lutes use movable frets made from gut string tied around the fingerboard, and so the frets are constrained to be continuous across the full width of the neck. You can try this yourself using 1 mm diameter nylon monofilament, as used for classical guitar G strings, if you learn the special knots for tying such slippery material tightly. But it appears that some fixed fretlets (*tastini* in Italian) were added, at least in the late 16th Century, to provide otherwise missing notes of an approximate meantone temperament.

Fretting or scale rotation

Now for each set of open-string intervals under consideration we draw a schematic diagram of a fretboard an octave and a half long, using an arbitrary scale rotation. That is, we choose some arbitrary note of the scale for the initial pitch of the open reference string and set the other strings to have the correct relationships with it, as specified by our state transition diagram. For convenience I use a spreadsheet column for each string where each row represents one step of some equal temperament which well-represents the linear temperament being used, but is large enough so that the scale under consideration is in no danger of wrapping around or closing on itself.

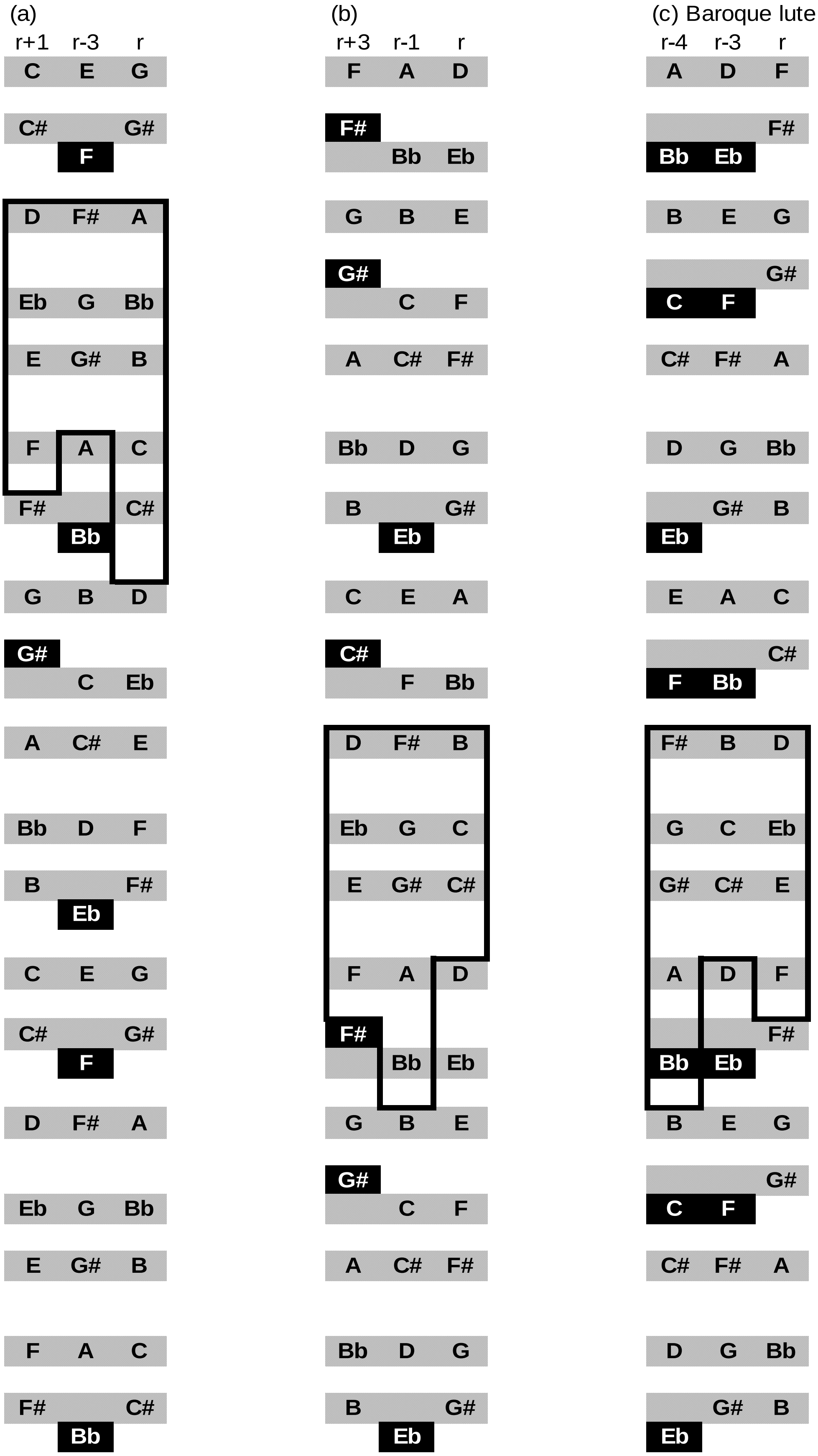


Figure 6. Fretting diagrams used to find the optimum scale rotation (nut or zeroth fret position).

Suitable equal divisions are listed in table 1 in the “Rep ET” column. I stress that this use of an ET is just a convenience in constructing these diagrams, and in the final result we should use an optimum generator for the linear temperament, which will not, in general, form a closed system.

For our 12-of-meantone example I’ve used 31-ET in constructing our fretboard diagrams, and in regard to the open tuning of figure 5(a) I’ve chosen G for our trial open reference-string. Remember that the reference or top string must have a fret for every pitch of the scale, so it determines the fretting for all the strings. Our perfect-fourth generator corresponds to 13 steps of 31-ET and so we can calculate the offset of each fret in steps of 31-ET from the offset of its pitch on the chain of generators, relative to the chosen open pitch. To do this we simply multiply the offset in generators by the size of the generator in steps of 31-ET (i.e. 13) and then reduce it modulo 31.

Pitch name	G#	C#	F#	B	E	A	D	G	C	F	Bb	Eb
Offset in generators	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4
Offset in steps of 31-ET	2	15	28	10	23	5	18	0	13	26	8	21

To obtain the actual fret positions in cents we would perform the same operation, but instead of multiplying by steps of 31-ET and reducing modulo 31, we would multiply by the size of our generator in cents (including any decimal fractions of a cent) and reduce modulo 1200. But we don’t need to do that until we’ve found the optimum open reference-string pitch.

Consider figure 6(a). Constructing this in a spreadsheet is simply a matter of laying out the first octave on the reference string according to the calculated offsets, and then copying and pasting to fill in the rest. In this case we don’t need to show all six strings since the other three are octave-equivalent to those shown. Then we colour the fret positions one colour (shaded) and the missing or unfretted notes (holes) another colour (black). Unlabelled positions on the frets correspond to extrascalar pitches. Then we determine the shape of the critical region (outlined) and in-effect slide this down the fretboard until we find a position where it does not include any missing notes, or at least minimises their number. The best such position is shown for the three cases in figure 6.

The pitches shown at the top of this optimally positioned critical region are the pitches of the open strings in the final design. Any missing notes that remain in the critical region, such as the Bb on the third string in 6(c), would need to be included by adding fretlets.

If there are several candidates having no missing notes *inside* their critical region, we would favour those whose missing notes are as far as possible *beyond* the critical region. In this regard the solution of 6(a) is superior to that of 6(b). The number of extrascalar notes in the critical region may also come into consideration as a tie-breaker in some cases.

It is possible that solutions 6(a) and 6(b) were discovered by baroque or renaissance lutenists and rejected for reason we have not yet considered. We have found that, when the same intervals are used, as between the open strings of the baroque lute, an absolute open tuning of F# B D F# B D is best for an Eb to G# meantone, so by transposition we can determine that the actual A D F A D F lute tuning was best suited for a Gb to B meantone, and in that case the missing note on the third and sixth strings would be Db. However, there is some evidence that baroque lutes were often tuned to something approaching 12-equal, in which case none of this matters.

But we now know how to make an Eb to G# meantone lute or guitar that has only 12 full-width frets per octave, by requiring that the open strings be tuned to either D F# A D F# A or D F# B D F# B.

I should now remind you that, to make the problem tractable, we have made a number of simplifying assumptions. For example we have ignored the complex details of what positions are actually possible for human fingers. And we have ignored the question of which notes may be used most often and so would benefit from being available on an open string. So before building such an instrument with fixed frets, one should check in some detail the actual fingering patterns for the various chords that one expects to play. This can be done, for example, by taping a notated paper template of the proposed fretting to the fretboard of an existing instrument and trying the required patterns. One should also check that frets, or fretlets on the same string, are no closer together than about 30 cents or it may be too difficult to position a finger between them, or to prevent the string buzzing on the higher fret when the lower is played.

Microtempering a JI scale

Now lets see what we can do with an actual JI scale. Let's use the 12-note 5-limit scale of the fretboard shown in figure one. Here it is shown on a lattice, first notated as comma inflected letter names and secondly as ratios. "\" can be read as "comma-down" and "/" as "comma-up".

```
      A\    E\    B\    F#\         5/3   5/4  15/8  45/32

   F     C     G     D           4/3   1/1   3/2   9/8

Db/   Ab/   Eb/   Bb/        16/15  8/5   6/5   9/5
```

By consulting table 1 we find that the simplest microtemperament for 5-limit is called kleismic and has a generator which is a minor third of 316.99 cents and a period which is a whole octave. It is *very* "micro", since it has a maximum error of only 1.4 cents.

So how do we apply this microtemperament to our just scale? The first thing we need to do is express each pitch of our scale as an ordered list of exponents of prime numbers. For example 9/5 can be expressed as $3^2 * 5^{-1}$, so its exponent list is [2, -1]. We can ignore the exponents of 2 since we are dealing with an octave-repeating scale. There is a simple relationship between the exponent list and the position on the lattice. The first number gives the horizontal position and the second number gives the vertical position.

```
        [-1 1]  [0 1]  [1 1]  [2 1]

    [-1 0]  [0 0]  [1 0]  [2 0]

[-1 -1] [0 -1] [1 -1] [2 -1]
```

We convert each pitch to a number of generators by multiplying each number in its exponent list by the corresponding number in the temperament's prime mapping and then adding the results together. From table 1 we find that the kleismic mapping from primes to generators is [6 5] (again ignoring octaves). For example, for the top-right pitch on the lattice we have 2*6 + 1*5 = 17 generators. Here's the entire lattice in terms of generators.

```
       -1     5    11   17

    -6     0     6    12

-11    -5     1     7
```

When this is laid out as a chain of minor third generators it is as follows. We see that unlike the meantone case, there are many gaps in the chain.

```
Db/        F Ab/     A\ C Eb/    E\ G Bb/    B\ D        F#\
-11 . . . .-6-5 . . .-1 0 1 . . . 5 6 7 . . .11 12. . . .17
```

We need to find the number of missing notes caused by various offsets in generators. To do this we slide two copies of the above chain against each other and count how many pitches on the offset chain now fall opposite spaces on the other chain. For example, we see below that an offset of 6 generators (corresponding to a fifth), results in only 3 missing notes.

```
-11 . . . .-6-5 . . .-1 0 1 . . . 5 6 7 . . .1112 . . . .17
            -11 . . . .-6-5 . . .-1 0 1 . . . 5 6 7 . . .1112 . . . .17
```

When we do this we discover that we only need to consider offsets of 5 or 6 generators, since every other offset has 6 or more missing notes. Strings designated as r-5 and r+5 have 4 missing notes while r-6 and r+6 strings have 3 missing notes, and of course r strings have zero missing notes.

Then we calculate the size in cents (modulo the octave) for various multiples of the generator, to see which are within the range 267 to 583 cents. We only need to consider those which are possible transitions between the abovementioned strings, i.e. ±1, ±5, ±6, ±10, ±11, ±12. We find that only transitions of 1, 5, -6 and -10 generators are within the allowable size range. The resulting state-transition diagrams are shown in figure 7.

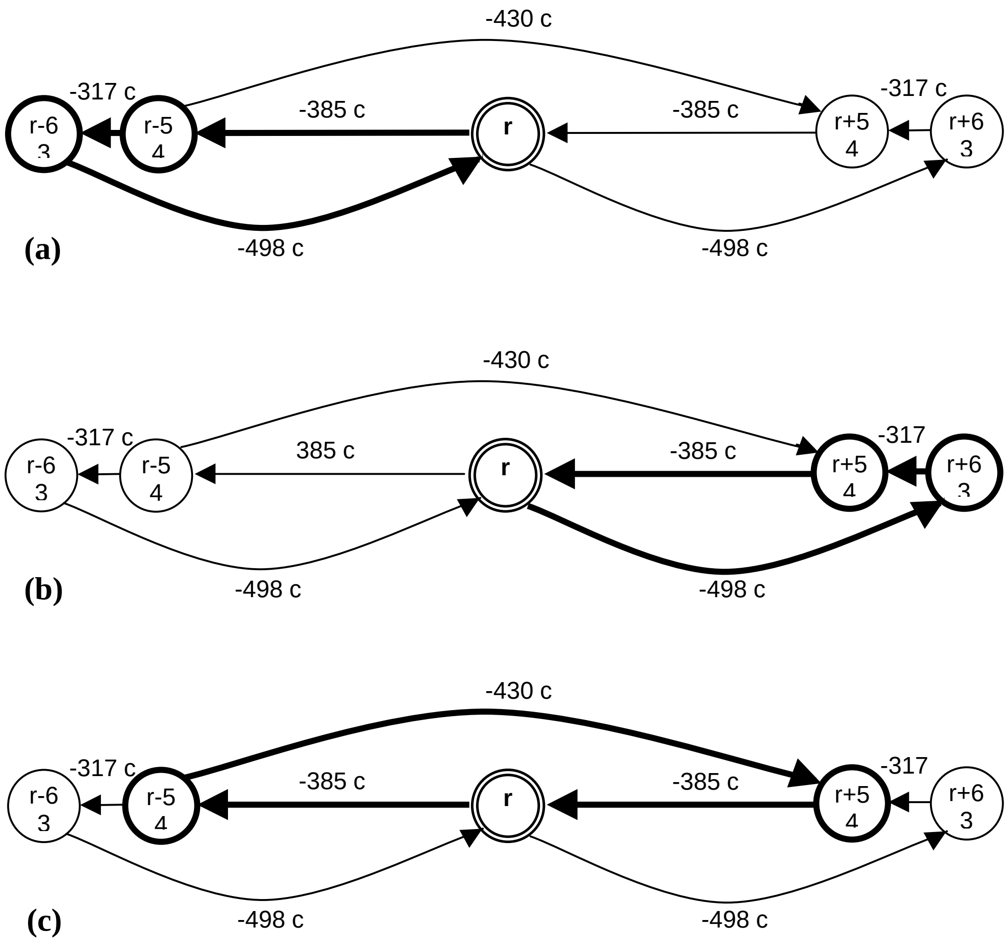


Figure 7. Kleismic tempered 12-of-5-limit-JI . State transition diagrams showing open string tunings and the intervals between them.

We see from the last column of table 1(a) that 14 steps of 53-ET is a suitable kleismic generator for constructing our fretboard diagrams. We start with the scale expressed as numbers of generators and multiply each by 14, modulo 53 to obtain the scale in terms of degrees of 53-ET.

```
Db/        F Ab/     A\ C Eb/    E\ G Bb/     B\ D      F#\
 5 . . . .22 36. . . 39 0 14. . .17 31 45. . .48 9. . . .26
```

Then we use these positions on the reference string to construct our fretboard diagrams, determine the shape of their critical regions, and find the optimum position for the nut or zeroth fret in each case. The results are shown in figure 8.

We find that all possible positions of the critical region have some missing notes (black). With open string tunings (a) and (b) however, there are a few positions that have only one missing note. The missing note of 8(a), Bb/ looks less likely to be missed than that of 8(b), A\. We could either accept this, or add the required fretlets, or we could try a 5-limit microtemperament with slightly larger errors.

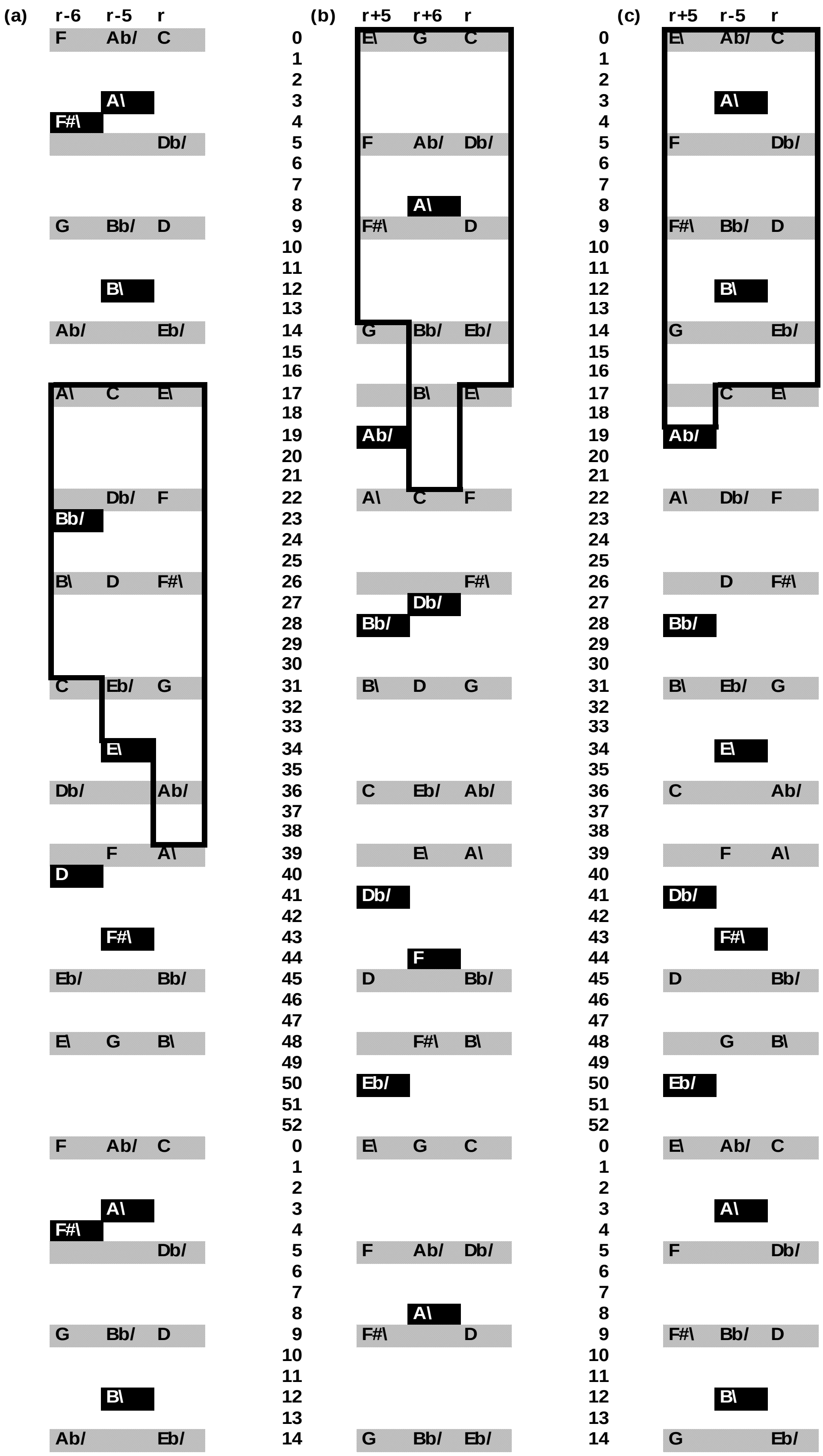


Figure 8. Kleismic tempered 12-of-5-limit-JI. Fretting diagrams.

Let's try diaschismic

In table 1(c) we find diaschismic with 3.3 cent errors, which has a standard generator of around 105.2 cents in two chains spaced a half octave apart. So instead of simply dealing with offsets in generators alone, which consist of a single integer, we must deal with offsets in generators and periods, as a pair of integers which we separate with a semicolon (generators;periods). In this case the second number is either 0 or 1, telling us which of the two parallel chains the pitch is in.

We start again with our pitches expressed as lists of exponents of odd primes, in this case 3 and 5.

```
        [-1 1]  [0 1]  [1 1]  [2 1]

    [-1 0]  [0 0]  [1 0]  [2 0]

[-1 -1] [0 -1] [1 -1] [2 -1]
```

Table 1(c) gives the mapping for diaschismic as [1 -2; 1 1]. The part before the semicolon gives the numbers of generators corresponding to each odd prime, and the part after the semicolon gives the numbers of periods for the same. Taking the pitch [-1 0] as an example we have -1*1 + 0*-2 = -1 generators and -1*1 + 0*1 = -1 periods. We could express this as -1;-1 but remember that periods should be expressed modulo the number per octave, in this case 2. So we express this pitch as -1;1. Here's the entire lattice in terms of generators and periods.

```
      -3;0  -2;1  -1;0   0;1

   -1;1   0;0   1;1   2;0

1;0    2;1   3;0   4;1
```

When this is laid out as two chains of semitone generators it looks like this.

```
A\     B\  C Db/ D Eb/
-3  . -1  0  1  2  3              ;0 chain
   -2 -1  0  1  2  .  4           ;1 chain
   E\  F F#\ G Ab/   Bb/
```

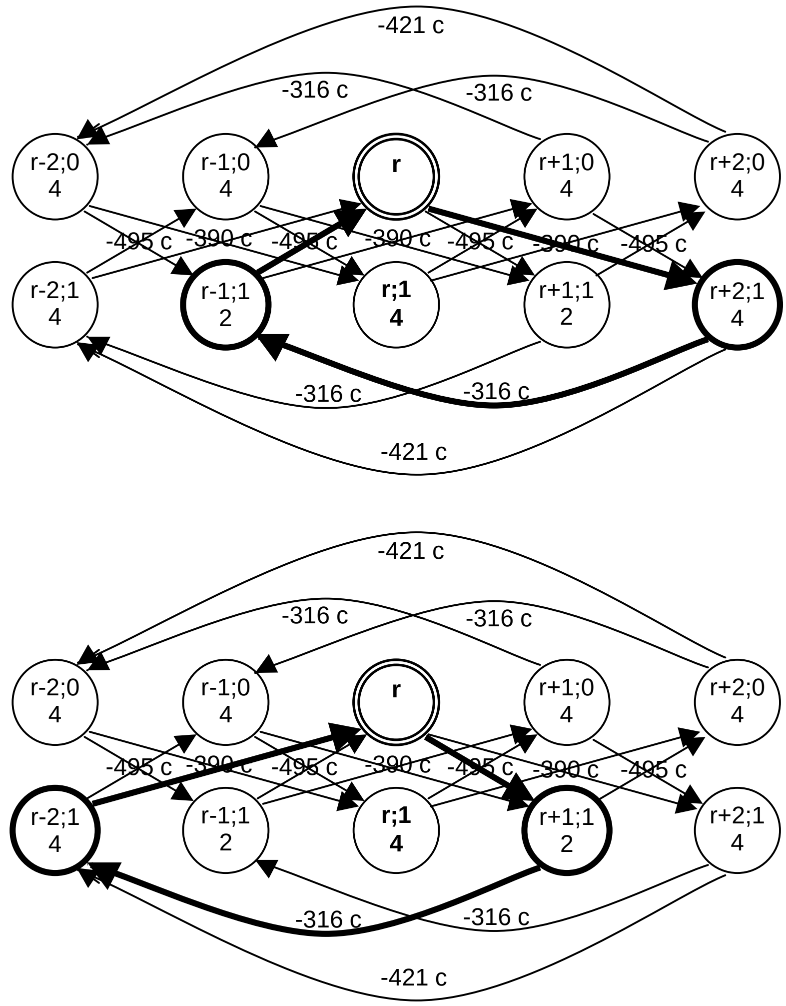
This time we need to find the number of missing notes caused by various offsets in generators *and* periods. This time we have two *pairs* of chains sliding past each other and possibly switching places. We can easily see from the above diagram that simply swapping the chains, corresponding to an offset of 0;1, will result in 4 missing notes. We can see below that an offset of 1;1 will result in only 2 missing notes.

```
-3  . -1  0  1  2  3               ;0 chain
      -2 -1  0  1  2  .  4            ;1 chain

   -2 -1  0  1  2  .  4            ;1 chain
   -3  . -1  0  1  2  3               ;0 chain
```

We find that we only need to consider open strings which have offsets of r+0;1 r+1;0 r-1;0 r+2;0 r-2;0 r+2;1 r-2;1 which have 4 missing notes and r+1;1 and r-1;1 which have only 2 missing notes, and of course r, the reference string itself, which has no missing notes by definition. All other offsets have 6 or more missing notes.

Then we calculate the size in cents (modulo the octave) for the various possible transitions between these strings, to see which are within the range 267 to 583 cents. We need to consider those in the ranges -4;0 to +4;0 and -4;1 to +4;1. For example +4;1 is 4*105 c + 1*600 c = 1020 c. The resulting state-transition diagrams are shown in figure 9.



(a)

(b)

Figure 9. Diaschismic tempered 12-of-5-limit-JI . State transition diagrams showing open string tunings and the intervals between them.

Table 1(c) tell us that 7 steps of 80-ET is a suitable generator to use for our fretboard diagrams for diaschismic, and of course the period will be 40 steps. So from the lattice showing generators and periods we can obtain the following lattice showing degrees of 80-ET.

```
      59    26    73    40            A\    E\    B\    F#\

   33     0    47    14            F     C     G     D

 7    54    21    68            Db/   Ab/   Eb/   Bb/
```

Unfortunately, when we plot these on fretboard diagrams for the two open tunings in figure 9, we find that we can do no better than the kleismic temperament, i.e. we still have at least one missing note in the critical region. So there is little point in accepting the larger error of diaschismic. We might have guessed if we had compared the complexities of the two temperaments, in the “Comp” column of table 1. Although the error went up, the complexity did not go down, but stayed at 6. But at least you got to see how to deal with a temperament whose period is a fraction of an octave.

Final results

So I will now give the precise tuning of one of the kleismic solutions above - the one with the open minor tuning A\ C E\ A\ C E\. We will use the more precise value of 316.99 c for the generator, but round the results to the nearest cent..

From figure 7(a), we see that the intervals between open strings are, ascending in pitch,

```
A\  317c  C  385c  E\  498c  A\  317c  C  385c  E\
```

For the fretting, we need to describe the tempered scale relative to the pitch of the open reference string, in this case an E\. We previously described the scale in terms of generators, which when sorted into pitch order starting from E\, is as follows.

```
E\  F  F#\  G  Ab/ A\  Bb/ B\  C  Db/  D   Eb/
5  -6  17   6  -5  -1   7  11  0  -11  12   1
```

We first subtract 5 from all the generator counts so they are relative to the E\, then we multiply them by the generator size of 316.99 c and octave reduce, to obtain the following fret tunings to the nearest cent..

```
nut fr1 fr2 fr3 fr4 fr5 fr6 fr7 fr8 fr9 fr10 fr11 fr12
  0 113 204 317 430 498 634 702 815 928 1019 1132 1200
```

We also need the tuning of the additional fretlets in the critical region on the A\ strings. From figure 8 we see that these are Bb/ which has an offset of 8 generators from the open A\, and so we find the fretlets are 136 c from the nut or zeroth fret.

Figure 10 shows a lute-style microtempered fretting where we have 12 full width frets (that could be implemented using securely tied nylon monofilament) plus the minimum fretlets to ensure there are no holes in the critical region. Note that the fretlets in figure 10 are only 23 cents up from the first fret. This is really too close for comfortable fingering and the first fret would be better turned into fretlets as it is in figure 11. If these fretlets had instead been close *below* the full fret, the issue would not be one of fingering (since we do not need to actually *play* the note on the full fret) but one of possible string buzzing.

We have not attempted to eliminate any extrascalar notes either inside or outside the critical region, so it might be wise to mark these somehow. For example, red self-adhesive dots might be placed above the offending fret positions, at least until their positions are learned. Nor have we attempted to fill any holes (missing notes) outside the critical region. In general, for every extrascalar note there is a nearby hole.

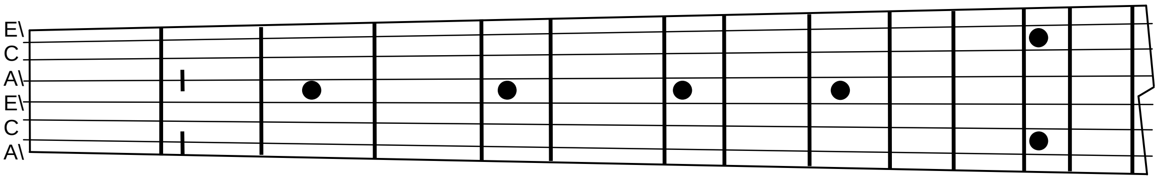


Figure 10. Lute-style microtempered 12-note 5-limit just intonation fretboard – minimum fretlets

Figure 11 shows a complete fretting that would ensure there were no holes and no extrascalar notes anywhere on the fretboard. These two frettings represent two extremes. Many intermediate solutions are of course possible.

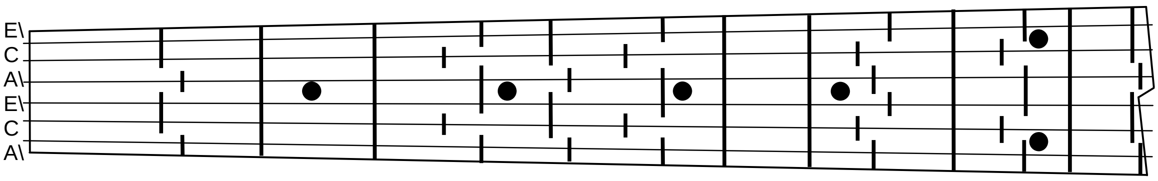


Figure 11. Max-fretlet-style microtempered 12-note 5-limit just intonation fretboard

We should also compare this to the strict JI tuning with the same open strings so we can see how much of any improvement is due to microtempering and how much is due simply to the choice of open tuning. When we do this, we find in this case, that any improvement is *entirely* due to the open string tuning, and there's actually no point in tempering it at all! That's because I chose a particularly simple scale to keep the examples manageable, and it is not well matched to the kleismic microtemperament. You will also find that the number of fretlets in figure 11 is the same as that in figure 1, although we have at least succeeded in moving most of them further from the nut.

If we had gone with the larger error of the diaschismic temperament we would have eliminated two fretlets from figure 11. Tunings having higher odd-limits, and more pitches, are likely to benefit more from this process, but the priority for this paper was to explain the method with simple examples. A first exercise for the interested reader might be to apply the method again to the example scale, this time using the wuerschmidt microtemperament of table 1(b).

Conclusion

This paper has presented some linear microtemperaments and has described one method of applying them to the optimisation of JI guitar fretboard designs. It is the only method I have so far devised for this, and I must admit it seems rather clumsy. I fully expect that future investigators will find ways to refine it. In particular, it seems rather crude to simply choose temperaments from a list, based only on the odd factors used in the scale, and ignoring the actual structure of the scale (e.g. as displayed on a lattice). But despite the apparent clumsiness of the method, the results it produces are often quite beautiful.

The method would clearly benefit from computerisation, which would allow many possibilities to be explored in a short time. In that case we could dispense with the first part of the method, where we use the state-transition diagram, and simply try all allowed open string tunings within the given representative ET.

I hope that some readers will look anew at some of their favourite JI scales, which may previously have seemed too difficult to implement on a fretted stringed instrument. And I invite them to report to the YahooGroup *tuning*, the results of applying the method, Feel free to contact me by email or mail if you need help in applying the method, or simply wish to discuss it.

Some other applications of microtemperaments not explored in this paper are:
(a) increasing the number of justly intoned consonances in a tuning, and
(b) eliminating the phase-locking that can make JI harmonies sound unnatural on some electronic instruments.

Both of these are automatic consequences of microtempering.

I am grateful to George Secor and Paul Erlich for their comments on an earlier draft of this paper, and to Paul Erlich and Carl Lumma for encouraging me to write it up in the first place.

Appendix – a *Blackjack* guitar

In case you felt cheated by the simplicity of the examples used above, and were hoping to come away from this paper with more than one non-trivial result of its application, I offer the following. (Note: I consider the continuous-fret meantone designs to be a non-trivial result of applying this method, although of course they are not the JI that was promised in the title.)

One means of ensuring a good match between a JI tuning and its microtemperament is to actually *start* with the microtemperament, i.e. start with contiguous pitches on chains of generators in some microtemperament. This is what we did for the (non-micro) meantone temperament example. Then if you wish, you can work backwards to find the JI tuning that this represents. In fact it will simultaneously represent more than one, but often one will make more sense than the others. The number of pitches used in the microtemperament should not be so large as to be overwhelming, and should be based on considerations of minimum fret-spacing and evenness, and would typically be *distributionally even*, a *periodicity block*, a *moment of symmetry*, or a *constant structure*. I won't go into the meaning of these terms here but you can refer to Joseph Monzo's wonderful tuning encyclopedia at http://www.tonalsoft.com/enc/, or Paul Erlich's *Gentle Introduction to Fokker Periodicty Blocks* at http://sonic-arts.org/td/erlich/intropblock1.htm or his *The Forms of Tonality* at http://lumma.org/tuning/erlich/erlich-tFoT.pdf.

One such tuning is that of Eduardo Sabat-Garibaldi's *Dinarra*. This is a guitar that uses the schismic microtemperament and may be considered 5-limit-plus-9s-and-15s (or 1.3.5.9.15-"limit"). See www.invention-ifia.ch/ifis/sectiong/g0101/g0101.htm. I personally feel that 53 frets per octave is excessive. I would have stopped at 29 frets even though the minimum fret spacing is the same; a rather daunting 21 cents.

Another microtemperament-derived tuning that has recently achieved some popularity in microtonal circles, particularly with the composer Joseph Pehrson, is the 21-of-miracle tuning, which Paul Erlich aptly named *Blackjack*.

One 7-limit *just*-ification of this tuning (with G as 1/1) is

```
1/1 21/20 15/14 9/8 8/7 6/5 128/105 5/4 21/16 4/3 7/5 10/7 3/2 32/21
8/5 105/64 12/7 7/4 64/35 15/8 63/32 (2/1)
```

You can see Paul Erlich's beautiful coloured 7-limit lattice diagram for this at http://groups.yahoo.com/group/tuning/files/perlich/scales/blackjust1.gif. The grey intervals on this diagram show the extraordinary number of additional 7-limit consonances made available by the microtempering.

You can learn more about Blackjack at http://sonic-arts.org/monzo/blackjack/blackjack.htm and you can hear some Blackjack pieces at http://www.soundclick.com/bands/5/josephpehrsonmusic.htm and http://www.soundclick.com/bands/5/tuninglabmusic.htm.

As far as I am aware, at the time of writing, no one has yet made a Blackjack guitar. Figure 12 shows the result of a straightforward application of the above method to this scale. There are only 3 missing notes on each non-reference string (which could be remedied with fretlets if you wish). There are no missing notes in the critical region and in fact each string has a fret corresponding to the open pitch of the next higher string. It is also good that it has the wider interval of a perfect fourth between the two lowest strings, and its range is almost that of a standard guitar, missing only ¾ of a tone at the low end and a whole tone at the high end.

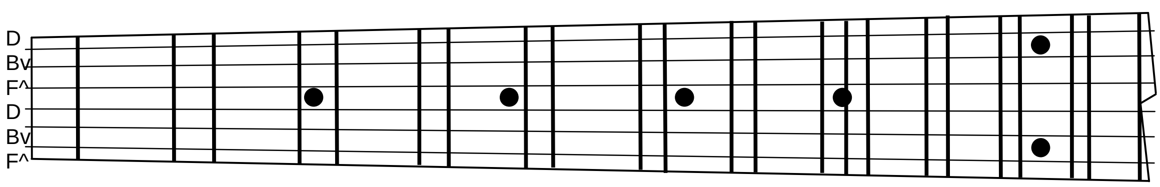


Figure 12. Blackjack fretboard

In this case, rather than use the optimum 7-limit miracle generator as per table 1(a), I have left it as 7 steps of 72-equal so that as many notes as possible coincide between Blackjack and 12-equal; namely the four notes CGDA. This is the *standard* Blackjack tuning as used by Joseph Pehrson. Even with this slightly-non-optimal generator, the 7-limit errors are less than 3.0 cents. The minimum fret spacing is 33 cents. This tuning also has many good approximations of 11-limit ratios, in particular neutral thirds and sixths. But I caution you again, to test the playability of the chords of interest, before committing to this, or any other design.

The intervals between open strings are, ascending in pitch,

```
F^  500c  Bv  350c  D  350c  F^  500c  Bv  350c  D
```

where `F^` is F half-sharp and `Bv` is B half-flat.

Here are the fret tunings to the nearest cent.

```
 nut  fr1  fr2  fr3  fr4  fr5  fr6  fr7  fr8  fr9 fr10
   0   33  117  150  233  267  350  383  467  500  583

fr11 fr12 fr13 fr14 fr15 fr16 fr17 fr18 fr19 fr20 fr21
 617  700  733  817  850  883  967 1000 1083 1117 1200
```

-- THE END --