Regular Temperament Theory: Exploring the Landscape between JI and ETs with Linear Algebra

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Abstract. Dissonance is easy. Consonance is rare. We want scales and tunings that give different flavors of consonance, without being too complex, and with small errors. Regular Temperament Theory (RTT) is a powerful tool, reducing the problem to approximating a few small prime numbers, and generating tunings by stacking a few intervals called generators. RTT opens a middle ground between just intonation and equal temperaments that goes far beyond meantone, applying linear algebra to discover new temperaments that balance complexity and error in different ways while providing harmonies that do not exist in conventional tunings. We introduce the theory and list some open problems.

Keywords: Microtonal, Temperament, Just Intonation

1 Introduction

This paper describes an application of linear algebra to an area of music theory called Regular Temperament Theory (RTT), whose purpose is to access novel harmonies based on ratios of small whole numbers up to 13 or so, without just intonation's complexity in number of notes, number of different step sizes, and difficulty of modulation. This comes at a cost in tuning accuracy. Various tradeoffs are possible.

Its development began around 1998 in a public online forum known as *the tuning list* [1], facilitated by Mills College, Oakland, California. The first author was involved from the beginning of that collaboration. Other major contributors to the theory were Paul Erlich [2], Graham Breed [3], Gene Ward Smith [4] and Mike Battaglia [5]. The theory has roots going back to the 1970s and earlier, with George Secor [6], Erv Wilson [7] and Adriaan Fokker [8]. Some composers who enriched this collaboration were Joe Monzo [9], Herman Miller, Margo Schulter, Dan Stearns, and Joseph Pehrson.

In January 2021 we set out to make the field more accessible. After surveying the existing resources, many including abstract algebra (group theory) and exterior algebra (wedge products), we laid out an approach covering all practical needs with linear algebra alone, giving examples in Wolfram Language. We replaced jargon and eponyms with consistent descriptive terms, and standardized unique variable letters and their styling. Finally, we reframed the basics of tuning optimization in favor of flexibility and real-world musical considerations, rather than mathematical purity or

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computational efficiency. This paper is a summary of the first 6 of 10 chapters in the resulting online textbook [10].

Milne et al summarized a part of this theory to generate keyboard layouts [11].

2 Mapping Just Primes to Temperament Generators

RTT allows us to generate new pitch systems (tunings) prior to scale generation and composition. The input vectors of this theory represent justly-intoned intervals modeled as frequency-ratios whose numerator and denominator are small positive integers. We represent them as *prime-count vectors* whose entries are integers that give the counts (or exponents) of successive primes in the prime factorization of the ratio, with negative counts for the denominator. For example, the interval of a just major third is a frequency ratio of 5/4 = 5/(2×2) = $2^{-2}\times3^{0}\times5^{1}$ which is represented as the column vector $\mathbf{i}$ = [-2 0 1]^{T}. We can call them "PC-vectors", but to avoid confusion with "pitch class vectors" we can simply call them "vectors" in RTT, where it is conventional to use a variant of Dirac's bra-ket notation. $\mathbf{i}$ = [-2 0 1⟩.

A row vector is a linear map. For example, the *just(-prime) tuning map* $\boldsymbol{j}$ = 1200 × $\log_2$(⟨2 3 5]) = ⟨1200.0 1902.0 2786.3] can be used to obtain the size of a just interval in cents. e.g. for the just major third, $\boldsymbol{j}\mathbf{i}$ = ⟨1200.0 1902.0 2786.3] [-2 0 1⟩ = 386.3 ¢.

A regular temperament is a temperament that can be represented by a rectangular integer matrix that corresponds to a linear mapping from prime-count vectors to integer vectors of lower dimensionality called *generator-count vectors* (GC-vectors). This matrix is called the temperament mapping matrix, or simply the *mapping*.

For example, 12-tone equal temperament (12-ET) is represented by the single-row mapping M = [12 19 28], which tells us that there is a single generator and that prime 2 is approximated by 12 generators, prime 3 by 19 generators, and prime 5 by 28 generators. From this we can deduce that the generator is an approximate semitone. This is confirmed when we map the just major third: $\mathbf{y}$ = $M\mathbf{i}$ = [12 19 28] [-2 0 1⟩ = [4⟩. We use a curved angle bracket to distinguish a GC-vector from a (PC-)vector. The single entry of 4 means the just major third is approximated by 4 generators, in this case 4 semitones.

We can think of the vector entries as having units of primes **p** and the GC-vector entries as having units of generators **g**, and so the mapping entries have units of generators per prime **g**/**p**.

We said the generator was an *approximate* semitone because this way of defining temperaments leaves open the exact tuning of the generators. That belongs to a separate optimization phase. All we need to know at this stage is that the generator is 1/12 of an approximate octave. We do not assume pure octaves or octave-equivalence.

Equal temperaments have mapping matrices with rank r = 1, and so are called *rank-1 temperaments*. So far, our input vectors have dimensionality d = 3 and are known as *5-limit* vectors (after the highest prime). A mapping matrix is always full-rank and so it is an $r \times d$ matrix.

Meantone is a well-known *rank-2 temperament*, dating from the early 1500s. It is conventionally described as having a generator that is a slightly-narrow perfect fifth, an approximate 3/2 ratio (~3/2). But from RTT's point of view, meantone has two generators, the first one being an approximate octave (~2/1). It is well known that meantone approximates a just major third (~5/4) using a stack of four fifths, octave-reduced. So it approximates prime 5 (2786.3 ¢) by a stack of four fifths (4 × ~700 = ~2800 ¢).

We can write the corresponding meantone mapping as $M = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4 \end{bmatrix}$. Looking at each column in turn, we see that prime 2 is approximated by one of the first generator, prime 3 by one of the first generator plus one of the second generator, and prime 5 by four of the second generator.

With more than one row, our mapping matrix is no longer unique. We can perform elementary integer row operations to find alternative mappings for meantone, associated with different generators. For example, we could use an octave and a perfect fourth (~4/3) as the generators. Since the fourth is the octave minus the fifth, we can perform the inverse operation on the rows of the mapping, adding the second row to the first and negating the second, to obtain $M = \begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & -4 \end{bmatrix}$.

This shows the need for a canonical form for mapping matrices, so we can recognize when different procedures give the same temperament. A first pass at a canonical form is to use Hermite normal form (HNF), integer analog of RREF. In Wolfram Language:

```
hnf[a_]:= Last[HermiteDecomposition[a]].
```

And we obtain $M = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 4 \end{bmatrix}$.

We can see that the first generator approximates prime 2 on its own, as before, so it is an approximate octave. The second generator approximates prime 3 on its own, so it is an approximate perfect twelfth (~3/1).

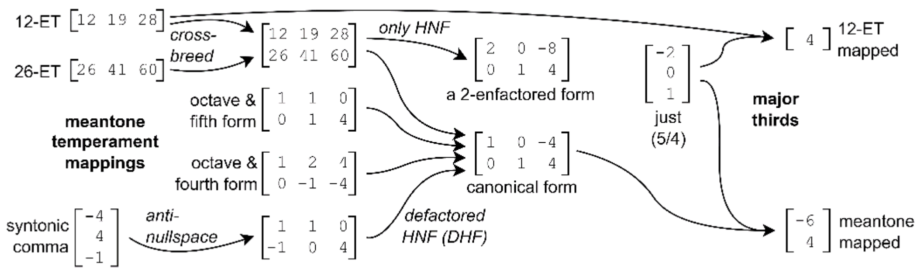

Fig. 1. Mapping just primes to temperament generators

We have so far obtained the meantone mapping by considering the generators. Another way is by "cross-breeding" equal temperaments. Both 12-ET and 26-ET can be considered as supporting (rather extreme) tunings of meantone. The 5-limit mapping for 26-ET is [26 41 60]. If we stack the two ET mappings, we have $M = \begin{bmatrix} 12 & 19 & 28 \\ 26 & 41 & 60 \end{bmatrix}$. Taking the HNF we obtain $M = \begin{bmatrix} 2 & 0 & -8 \\ 0 & 1 & 4 \end{bmatrix}$.

The latter *would* be the canonical meantone mapping except for a common factor of 2 in the top row. We say the matrix is *enfactored*. This is musically wasteful as half of the generator count vectors would not correspond to the tempering of any interval. Note that the common factor is hidden in the original matrix. The HNF may not reveal the common factor either. It may only become apparent in some far-from-obvious linear combination of the rows, using coprime integer multipliers (if not coprime, they would be *introducing* rather than *revealing* a common factor). A simple defactoring (or saturation) algorithm, which works even in these hidden cases, consists in performing a column-wise Hermite decomposition, keeping only the unimodular matrix, inverting it and taking the top r rows, where r is the rank. Taking the HNF of that yields our final canonical form which we call *defactored Hermite form* (DHF). In Wolfram Language:

```
dhf[m_]:= hnf[Take[Inverse[Transpose[First[
HermiteDecomposition[Transpose[m]]]]], MatrixRank[m]]]
```

Meantone equates the tones 10/9 and 9/8, hence the name. This means that their quotient, the syntonic comma 81/80 = [-4 4 -1⟩, is in the nullspace of the mapping, and since nullity is dimensionality minus rank, $n = d - r = 1$, this comma alone forms a basis for the nullspace. So the mapping matrix can be obtained as the *dual* of the comma basis matrix [[-4 4 -1⟩], which in this case has only one column (vector). If this was a 7 limit (4D) rank-2 temperament, a second comma would vanish, such as 225/224, and the comma basis would have two columns [[-4 4 -1 0⟩ [5 -2 -2 1⟩]. In Wolfram Language:

`dualMapping[c_]:= dhf[NullSpace[c]]` (NullSpace is its own inverse)

When we apply this to the comma basis consisting only of the syntonic comma, we again obtain the canonical meantone mapping.

If we apply the canonical meantone mapping to the vector for the just major third we obtain the GC-vector $\boldsymbol{y} = M\mathbf{i}$ = [-6 4⟩. This says that the meantone-mapped major third consists of 4 of the approximate twelfth generators minus 6 of the approximate octave generators. The concepts covered in this section are summarized in Fig 1.

3 Tuning Optimization

Now for the optimization phase. We seek a *generator tuning map* $\boldsymbol{g}$ = ⟨~1200 ~1900] whose entries are optimal generator sizes. With $\boldsymbol{g}$ we could obtain the size of any tempered interval in cents as $\boldsymbol{g}\mathbf{y} = \boldsymbol{g}M\mathbf{i}$ analogous to obtaining the size of a just interval as $\boldsymbol{j}\mathbf{i}$. Therefore, the *error* in cents of a tempered interval is $\mathrm{e} = \boldsymbol{g}M\mathbf{i} - \boldsymbol{j}\mathbf{i} = (\boldsymbol{g}M - \boldsymbol{j})\mathbf{i}$.

An optimal $\boldsymbol{g}$ would minimize the audible harm done to the just intervals we care about. For a single interval we model this harm as the *damage* d, defined as the absolute value of the error multiplied by some positive *weight* w, so $\mathrm{d} = |\mathrm{e}|w$. The weight is typically one of three things: unity (unweighted) $w = 1$, or some measure of the *complexity* of the interval $w = c$, or the reciprocal of a complexity, called a *simplicity*, $w = 1/c$. The interval complexity measure is typically some pre-scaled power-norm $c = \|X\mathbf{i}\|_q$ where X is a diagonal (scaling) matrix. The most common complexity measure scales each entry by the base-2 log of its prime, then takes the taxicab norm: $X = \mathrm{diag}(\log_2([2\ 3\ 5\ \ldots]))$, $q = 1$. This is called the *log product* (lp) complexity as it is equal to the log of the product of the numerator and denominator of the corresponding ratio.

The *target-interval list* T is the list of intervals whose damage we care about; the fewer, the less damaged each can be. It may be chosen from a type of music, or the intervals playable on an instrument, but a useful default is the *truncated integer-limit triangle* (TILT). This begins as a triangular table of the ratios >1 between the positive integers less than the prime after the temperament's prime limit; for the 5-prime-limit this is the 6-integer-limit and for the 7-prime-limit it is the 10-integer-limit. We then remove intervals smaller than 15/13 (247.7 ¢) and larger than 13/4 (2040.5 ¢). See [10].

So for 5-limit meantone we suggest the 6-TILT, which consists of the eight intervals 2/1, 3/1, 3/2, 4/3, 5/2, 5/3, 5/4, 6/5, converted to prime-count vectors and placed side-by-side to form a target-interval list T, a $d \times k$ matrix where k is the number of target intervals. See Fig. 2.

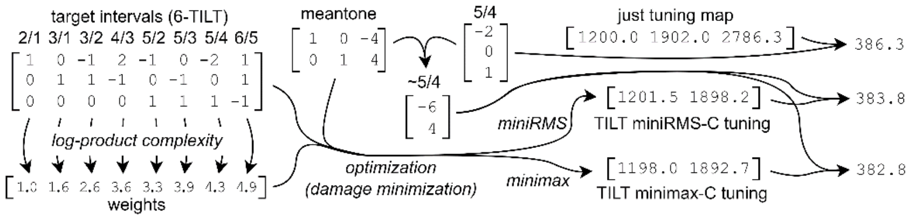

Fig. 2. Tuning optimization

We then compute the *target-interval error list* as $\mathbf{e} = (\boldsymbol{g}M - \boldsymbol{j})\mathrm{T}$, and for this run we'll compute the *target-interval weight list* $\boldsymbol{w}$ as the log product complexities of the target intervals. We then turn the weight list into a diagonal weight matrix $W = \mathrm{diag}(\boldsymbol{w})$ and compute the *target-interval damage list* as $\mathbf{d} = |\mathbf{e}|W$, where $|\cdot|$ is entry-wise absolute value. In terms of the generator tuning map $\boldsymbol{g}$, the damage list is $\mathbf{d} = |(\boldsymbol{g}M - \boldsymbol{j})\mathrm{T}W|$.

Popular statistics for the overall damage are the maximum damage $⟪\mathbf{d}⟫_\infty$ and the RMS damage $⟪\mathbf{d}⟫_2$, and therefore the popular optimization procedures are minimax and miniRMS (least squares). The double-angle-brackets are our notation for power-means, by analogy with power-norms.

MiniRMS has the advantage of a simple closed-form solution for $\boldsymbol{g}$ involving the Moore-Penrose inverse, $\boldsymbol{g} = \boldsymbol{j}\mathrm{T}W(M\mathrm{T}W)^{+}$. In Wolfram Language:

```
g = j.t.w.PseudoInverse[m.t.w]
```

The result is $\boldsymbol{g}$ = ⟨1201.5 1898.2]. We call this the TILT miniRMS-C tuning for meantone. The "C" stands for "complexity weight". To obtain the size of the more conventional generator, the fifth, we can subtract the first generator from the second to obtain 696.7 ¢. For reference, $\mathbf{e}$ = [1.5 -3.8 -5.3 6.8 -1.2 4.1 -2.6 -2.6] ¢.

A minimax value for $\boldsymbol{g}$ can be found using: `MaxIterations->100;`

```
NMinimize[ Max[Abs[(g.m-j).t.w]], g∈Vectors[2,Reals] ]
```

And we find that the TILT minimax-C tuning for meantone is $\boldsymbol{g}$ = ⟨1198.0 1892.7]. So the fifth is 694.7 ¢, and $\mathbf{e}$ = [-2.0 -9.3 -7.2 5.2 -5.4 1.8 -3.4 -3.8] ¢.

The minimax tuning found by this method is often not unique. In such cases, what we really want is the limit of the $\boldsymbol{g}$s that minimize the p-mean of the damages $⟪\mathbf{d}⟫_p$ as p approaches infinity. We can use the following Wolfram Language code, replacing the "2" with successive powers of 2, until the generators change by less than, say, 0.1 ¢. Numeric precision may need to be increased to avoid spurious results.

```
NMinimize[ Total[((g.m-j).t.w)^2], g∈Vectors[2,Reals] ]
```

We can also include constraints, such as holding some intervals unchanged by the tuning — typically the octave. A list of vectors can be assembled into a *held-interval basis* H, applying the constraint that $\boldsymbol{g}M\mathrm{H} = \boldsymbol{j}\mathrm{H}$. In Wolfram Language:

```
NMinimize[ { Total[((g.m-j).t.w)^2], g.m.h==j.h }, …]
```

4 Conclusion

Although we used meantone as a familiar example, an extraordinary example is George Secor's *miracle* temperament [6], $M = \begin{bmatrix} 1 & 1 & 3 & 3 & 2 \\ 0 & 6 & -7 & -2 & 15 \end{bmatrix}$. Its held-octave 12-TILT minimax-U generators are $\boldsymbol{g}$ = ⟨1200.0 116.72] ¢ with a maximum unweighted damage of 3.32 ¢. A 21-note-per-octave miracle scale has only two step sizes, contains many melodic subset scales, and supports a network of 11-(prime)-limit harmony that would require many times that number of notes in JI. Its rediscovery in 2001 started a "gold rush" that led to the development of the tools described above, and many others. Searches were conducted both by combining commas and by combining ETs.

Open problems regarding temperaments include: staff notation, family classification, anti-JI intervals [12], refining search criteria (including complexity × damage = badness measures), and psychoacoustic experiments to validate them.

Many temperaments have been cataloged in the Xenharmonic Wiki, where you will also find many details that were omitted from this paper due to space limitations, in our extended exposition entitled *D&D's Guide to RTT* [10].

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